Analysis

with ultrasmall numbers

Standard Level

PART I

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> *Infinity itself looks flat and uninteresting. [...] The chamber [...] was anything but infinite, it was just very very very big, so big that it gave the impression of infinity far better than infinity itself.*
> *(Douglas Adams: The Hitchhiker's Guide to the Galaxy)*

1

Introduction

1.1 Velocity and Position

Exercise 1

Suppose the velocity [1] of a car is constant and equal to $60km/h$.

(1) Let f be the function which describes the position of the car with respect to time. Draw the graph f for t ranging from 0 to 3 hours.

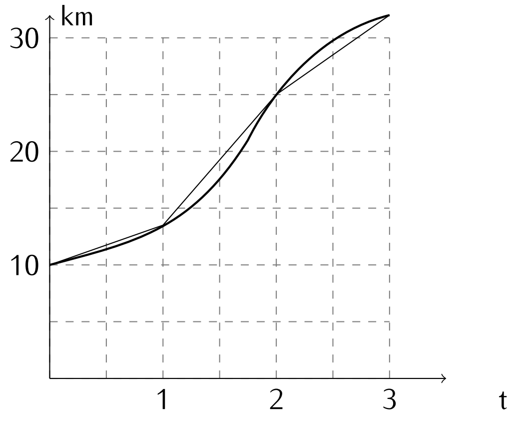
(2) Let v be the function which describes the velocity of the car with respect to time. Draw the graph of v for t ranging from 0 to 3 hours.

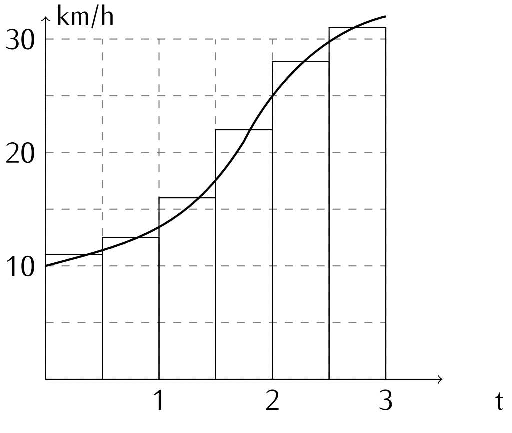
(3) Given the position graph, how can one find the velocity of the car at any given time?

(4) Given the velocity graph, how can one find the position of the car after any given time?

⚠ Note the difference: velocity (deduced from position) is *local*. It is possible to give the velocity *at* a given time. Position (deduced from velocity) is *global*. It is only possible to find the *variation* of the position over an *interval* of time.

A curve can be approximated by a piecewise linear function whose slope is easily calculated by pieces. It can also approximated by a "staircase" function whose area is calculated by adding the areas of the rectangles.





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[1] The velocity is speed with a direction. Speed is always positive (or zero); velocity can be negative.

The main goal of the subject called **mathematical analysis** will be to check when and how to approximate a curve by pieces of straight lines and when and how to approximate areas by rectangles and to understand what these can be used to calculate. Intuitively, it should seem clear that in order for the approximation to be good, the pieces of straight lines or the rectangles must be small – or that the number of pieces is large. The crucial questions are: How small? and How large?

1.2 Tiny and Huge

Exercise 2

If δ is a positive value which is extremely small (even smaller than that!),

(1) what can you say about the size of δ^2, $2 \cdot \delta$ and $-\delta$?

(2) what can you say about $2 + \delta$ and $2 - \delta$?

(3) what can you say about $\frac{1}{\delta}$?

Exercise 3

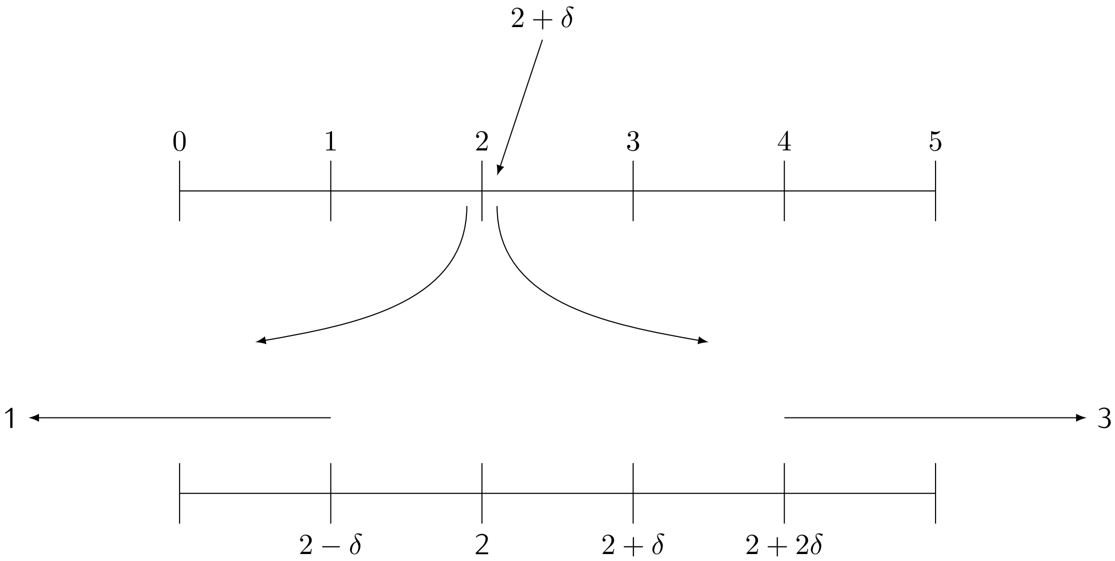
If N is a positive huge number (really very huge!),

(1) what can you say about N^2, $2N$ and $-N$?

(2) what can you say about $N + 2$ and $N - 2$?

(3) what can you say about $\frac{1}{N}$?

(4) what can you say about $\frac{N}{2}$?



Zooming in

Exercise 4

Let $f : x \mapsto x^2$, and let δ be vanishingly small and positive.

(1) Draw the result of a zoom centred on $\langle 1;1\rangle$ of f so that δ becomes visible.
Show, on the drawing, the values 1 and $f(1)$, $1+\delta$ and $f(1+\delta)$, $1-\delta$ and $f(1-\delta)$.
What does the curve look like?

(2) For the same function, draw the result of a zoom centred on $\langle 2;4\rangle$
Show, on the drawing, the values 2 and $f(2)$, $2+\delta$ and $f(2+\delta)$, $2-\delta$ and $f(2-\delta)$.

(3) Similar question for a zoom centred on $\langle 0;0\rangle$.

(4) Similar question for a zoom centred on $\langle -1;1\rangle$.

Exercise 5

Draw the result of zooms so that a tiny δ becomes visible for
$g : x \mapsto x^3$, and $h : x \mapsto |x|$
For g: centres are $\langle 1;1\rangle$, $\langle -2;-8\rangle$ and $\langle 0;0\rangle$
For h: centres are $\langle 1;1\rangle$, $\langle -2;2\rangle$ and $\langle 0;0\rangle$

Exercise 6

Draw a zoom centred on $\langle 0;0\rangle$ and another zoom centred on $\langle 0;-1\rangle$ for

$$k : x \mapsto \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

When we say that δ is "tiny", we want it to be tiny compared to all the parameters involved; this leads to the following definition:

Definition 1

The ***context*** *of a property, function or set is the list of parameters used in its definition. The context can be a single number.*

A context is *extended* if parameters are added to the list.

Before defining more precisiely what it means to be "tiny" we must first clarify what it means to be observable:

Observability Principle

- A number is observable relative to a context if it is observable relative to at least one parameter of the context.
- Every number is observable relative to some context.
- Two numbers a and b will always have a common context. If a is not observable relative to b, then b will be observable relative to a.

The word "observable" , by convention, refers to a context. Informally: the context is the parameters, sets and functions the statement is about. Therefore to determine the context of a statement, one must be able to understand it and describe what it says and about what it says something.

If a number is observable whenever any other number is observable, we say that it is *always* observable. We may also say that it is *standard*.

The following principle states that when we combine observable numbers, the result is also observable.

Closure Principle
Numbers, sets or functions, defined without reference to observability are always observable. If an object satisfies a given property, then there is an observable object satisfying that property

In the last sentence the context of observability is given by the property. There can be several values satisfying a given property, but at least one is observable and if there is only one number satisfying a given property, then this one is observable. Numbers that are always observable can also be called *standard*.

All "familiar" numbers such as $1;\ 3;\ 10^{10};\ \sqrt{2}$ or π are always observable, or standard, but also – in general –

$$\boxed{f(a)\quad\text{is observable}}$$

This refers to the context, by the word "observable". The only parameters of this property are f and a. This is the context.

Non observable values do not show up unless explicitly summoned.

Definition 2
A real number is ***ultrasmall*** *if it is nonzero and smaller in absolute value than any strictly positive observable number*

This definition makes an implicit reference to a context.

⚠ Note that 0 is not ultrasmall.

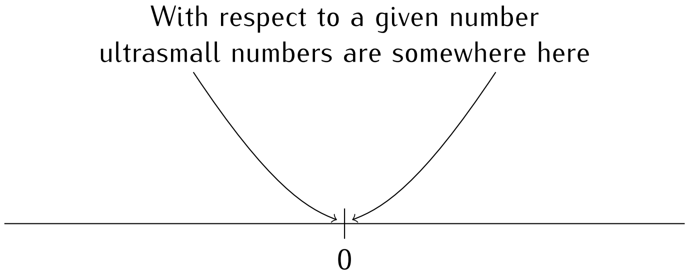
Principle of ultrasmallness
Relative to any context, there exist ultrasmall real numbers.

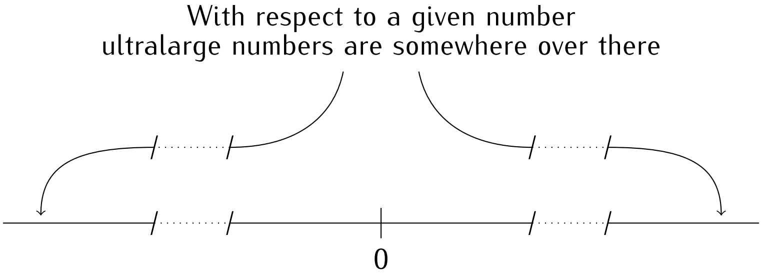
Such an ultrasmall number is then part of an **extended context**.

Given a context, if ε is ultrasmall then ε is not observable.

Definition 3
A real number is ***ultralarge*** *if it is larger in absolute value than any strictly positive observable number*

⚠ Note the asymmetry: if h is ultrasmall relative to x, then it is not observable. But x is observable relative to h (see the third item of the observability pricniple), hence x **is not** ultralarge relative to h.





Definition 4
*Let a, b be real numbers. We say that a is **ultraclose** to b, written*

$$a \simeq b,$$

if $b - a$ is ultrasmall or if $a = b$.

This definition makes an implicit reference to a context.
In particular, $x \simeq 0$ if x is ultrasmall or zero.

If $a \simeq b$ then a and b are said to be neighbours. If a is a neighbour of b and is observable (relative to some context) then a is the observable neighbour of b.

A rational number may have an observable neighbour which is not rational. The number $\sqrt{2}$ is always observable because it is completely and uniquely defined by the parameter 2. Relative to this context consider an ultralarge N and take the first N digits of $\sqrt{2}$. This is a rational number which is not observable. Yet it is ultraclose to an observable number which is $\sqrt{2}$.

The existence of an observable neighbour is given by the following

Principle of the observable neighbour
Relative to a context, any real number x which is not ultralarge can be written in the form $a + h$ where a is observable and $h \simeq 0$.

Exercise 7
Show that if x has an observable part, then it is unique.

This unique number is **the observable neighbour** of x.

Exercise 8

Prove the following:

Theorem 1
Let $[a;b]$ be an interval. Show that if x is in $[a;b]$, then the observable part of x is not outside $[a;b]$.

Exercise 9

Prove the following:

(1) If ε is ultrasmall relative to x then $\frac{1}{\varepsilon}$ is ultralarge relative to x.

(2) If M is ultralarge relative to x then $\frac{1}{M}$ is ultrasmall relative to x.

Exercise 10

Prove the following theorems (together they give all the rules needed for analysis and will be referred to by "ultracomputation" or "ultracalculus"):

Theorem 2
Let ε and δ be ultrasmall relative to a context and let a be observable.

(1) Then: $a \cdot \varepsilon$ is ultrasmall.

(2) Then: $\varepsilon + \delta \simeq 0$

(3) Then: $\varepsilon \cdot \delta$ is ultrasmall

(4) If $a \neq 0$ Then: $\dfrac{a}{\varepsilon}$ is ultralarge

Theorem 3 (Ultracomputation)
Relative to a context: If a and b are observable and if $a \simeq x$ and $b \simeq y$,

(1) $a + b \simeq x + y$

(2) $a - b \simeq x - y$

(3) $a \cdot b \simeq x \cdot y$

(4) If also $b \neq 0$, $\dfrac{a}{b} \simeq \dfrac{x}{y}$.

For the last item of theorem 8, it is enough to show

Theorem 4
Relative to a context. If b is observable and $b \neq 0$ and if $b \simeq y$ then $\dfrac{1}{b} \simeq \dfrac{1}{y}$

and use item 3 to conlcude.

Practice exercise 1 Answer page 10

Consider a context.

(1) Give an example of x and y such that $x \simeq y$ but $x^2 \not\simeq y^2$.

(2) Give an example of x and y such that $x \simeq y$ but $\frac{1}{x} \not\simeq \frac{1}{y}$.

Practice exercise 2 Answer page 10

Relative to a context.

In the following, assume that ε, δ are positive ultrasmall and H, K positive ultralarge numbers. Determine whether the given expression yields an ultrasmall number, an ultralarge number or a number in between.

(1) $1 + \dfrac{1}{\varepsilon}$

(2) $\dfrac{\sqrt{\delta}}{\delta}$

(3) $\sqrt{H+1} - \sqrt{H-1}$

(4) $\dfrac{H+K}{H \cdot K}$

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7}$

(6) $\dfrac{\sqrt{1+\varepsilon}-2}{\sqrt{1+\delta}}$

Practice exercise 3 Answer page 11

Relative to a context find ultrasmall ε and δ (or the relation between them) such that $\dfrac{\varepsilon}{\delta}$ is:

(1) not ultralarge and not ultrasmall,

(2) ultralarge,

(3) ultrasmall.

⚠ The previous exercise show that if no relation is known between ultrasmall numbers ε and δ, their quotient can be of any possible magnitude.

Contextual Notation

The only acceptable properties are those that do not refer to observability or those that use the symbol "$\simeq$".

Answers to practice exercises

Answers to practice exercice 1, page 9

(1) Let $x = N$ be ultralarge, and $y = N + \frac{1}{N}$ so $x \simeq y$ but $x^2 = N^2 \not\simeq N^2 + 2 + \frac{1}{N^2} = y^2$.

(2) Let h be ultrasmall, then let $x = h$ and $y = h^2$. Then $x \simeq 0$ and $y \simeq 0$ hence $x \simeq y$. Then $\frac{1}{h}$ and $\frac{1}{h^2}$ are both ultralarge and $\frac{1}{h^2} - \frac{1}{h} = \frac{1}{h}(\frac{1}{h} - 1)$. By ultracomputation, this is ultralarge, hence $\frac{1}{x} \not\simeq \frac{1}{y}$.

Answers to practice exercice 2, page 9

The terms ultrasmall or ultralarge all refer to a given context.

(1) As $\frac{1}{\varepsilon}$ is ultralarge $1 + \frac{1}{\varepsilon}$ is ultralarge.

(2) We have $\frac{\sqrt{\delta}}{\delta} = \frac{1}{\sqrt{\delta}}$ which is ultralarge.

(If $\delta < c$ for any observable c, then $\sqrt{\delta} < \sqrt{c}$ and $\sqrt{\delta} \simeq 0$ hence $\frac{1}{\sqrt{\delta}}$ is ultralarge.)

(3) Maybe surprisingly, this is ultrasmall. To see this we multiply and divide by the conjugate:

$$\begin{aligned}\sqrt{H+1} - \sqrt{H-1} &= \frac{(\sqrt{H+1} - \sqrt{H-1})(\sqrt{H+1} + \sqrt{H-1})}{\sqrt{H+1} + \sqrt{H-1}} \\ &= \frac{(H+1) - (H-1)}{\sqrt{H+1} + \sqrt{H-1}} \\ &= \frac{2}{\sqrt{H+1} + \sqrt{H-1}}.\end{aligned}$$

H is assumed positive, its square root (plus or minus 1) is also a positive ultralarge. The sum of 2 positive ultralarge numbers is ultralarge hence the quotient is ultrasmall.

(4) $\dfrac{H+K}{HK} = \dfrac{1}{K} + \dfrac{1}{H}$ is ultrasmall.

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7} = \dfrac{35 + 7\varepsilon - 35 - 5\delta}{49 + 7\delta} = \dfrac{\overbrace{7\varepsilon - 5\delta}^{\simeq 0}}{\underbrace{49 + 7\delta}_{\simeq 49}}$ is ultrasmall or zero.

(6) $\dfrac{\overbrace{\sqrt{1+\varepsilon} - 2}^{\simeq -1}}{\underbrace{\sqrt{1+\delta}}_{\simeq 1}} \simeq -1$, hence not ultralarge and not ultrasmall.

Answers to practice exercice 3, page 9

(1) Take $\varepsilon = \delta$ then $\dfrac{\varepsilon}{\delta} = 1$.

(2) Take $\delta = \varepsilon^2$, then $\dfrac{\varepsilon}{\delta} = \dfrac{1}{\varepsilon}$ is ultralarge.

(3) Take $\varepsilon = \delta^2$, then $\dfrac{\varepsilon}{\delta} = \delta$ is ultrasmall.

2

Derivatives

We will often use dx to indicate an ultrasmall *increment*[1] of the variable x. It may be positive or negative but will never be chosen to be 0.

Exercise 11

Let

$$f : x \mapsto x^2$$

The graph of this function is a curve (a parabola). Zoom in on the point $\langle 2, 4\rangle$. 2 and 4 are always observable. Consider the value of the function at $2 + dx$, (for dx ultrasmall as mentioned above) and draw a straight line passing through $\langle 2, 4\rangle$ and $\langle 2 + dx, f(2 + dx)\rangle$.

- What is the slope of this straight line?
- What observable value is this slope ultraclose to?

Definition 5

A real function f defined on an interval containing a is ***differentiable at*** *a if there is an observable value D such that*

$$\frac{f(a + dx) - f(a)}{dx} \simeq D$$

not depending on dx.
Then $D = f'(a)$ is the ***derivative*** *of f at a.*

When the derivative exists, it is the observable neighbour of $\dfrac{f(a + dx) - f(a)}{dx}$.

Metaphorically, finding the derivative can be described by: Zoom in. If what you see is indiscernible from a straight line, then measure the slope of that line. Zoom out. Drop what you can't see.

Exercise 12

Using definition 5 calculate the derivative of: $f : x \mapsto 3x^2 + x - 5$ at $x = -2$ and $x = 2$.

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[1] increment: a positive or negative change in a variable.

Exercise 13

Using definition 5 calculate the derivatives (if they exist) of the following:

(1) $g : x \mapsto 2x^3 - 2$ at $x = 1$ and $x = 0$.

(2) $h : x \mapsto |x|$ at $x = 2$, $x = -2$ and at $x = 0$.

Exercise 14

Let $f : x \mapsto x^3 - 3x - 2$. Check that 2 is a root of f. Are there other roots?

At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

Exercise 15

Let $f : x \mapsto 2x^3 - 4x^2 + 2x$. At what values of x is the function equal to zero? At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

Practice exercise 4 Answer page 21

Calculate the derivative of the following:

(1) $f : x \mapsto 5x^2 - 10x$ at $x = 2$

(2) $g : x \mapsto 5(x - 10)^2$ at $x = 3$

(3) $h : x \mapsto x^4 + x^3 + x^2 + x + 1$ at $x = 1$

(4) $k : x \mapsto 5x^2 + 10$ at $x = 2$

Exercise 16

Consider the derivative at x (general case) of the function

$$f : x \mapsto x^2 + 3x.$$

Show that it is differentiable for all x and that $f'(x) = 2x + 3$.

Notice that in a derivative, if there is one, the division is **always** between two ultrasmall numbers. They <u>cannot</u> be replaced by 0 since $\frac{0}{0}$ is not defined.

3

Continuity

Informally: a function is continuous if it is where you would expect it to be by observing where it is just before and just after.

Definition 6 (Continuity)
Let f be a real function defined around a. We say that f ***is continuous at*** *a if*

$$x \simeq a \Rightarrow f(x) \simeq f(a).$$

The continuity of f at a is a property of f and a. Hence the context is given by f and a.

Exercise 17
Show that $f : x \mapsto x^3$ is continuous at $a = 2$.

Exercise 18
Show whether $f : x \mapsto \dfrac{x}{x^2+1}$ is continuous for all values of x.

Exercise 19

(1) Show that $f : x \mapsto |x|$ is continuous at $x = 0$, at $x = 1$, at $x = -1$ and at x in general.

(2) Show that $g :\mapsto \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$ and at x in general.

(3) Show that $g :\mapsto \begin{cases} x^2 & \text{if } x \geq -1 \\ x^3 & \text{if } x < -1 \end{cases}$ is not continuous at $x = -1$ but is continuous for all other values of x.

Exercise 20
Prove the following theorem:

Theorem 5
Let f and g be two real functions continuous at a. Then

(1) $f \pm g$ is continuous at a.

(2) $f \cdot g$ is continuous at a.

(3) $\dfrac{f}{g}$ is continuous at a if $g(a) \neq 0$.

Exercise 21

Prove the following theorem:

Theorem 6

Let f and g be two real functions. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

A function f is defined on the left of a (resp. on the right) if $f(x)$ is defined for all $x \simeq a$ with $x < a$ (resp. $x > a$). It is clear that f is defined around a if and only if f is defined on the right and on the left of a.

We now extend the concept of continuity at a point to continuity on an interval.

Definition 7 (Continuity on an Interval)

*(1) Let f be a real function defined on the open interval $]a;b[$. Then f is **continuous on** $]a;b[$ if f is continuous at all $x \in]a;b[$.*

*(2) Let f be a real function defined on the closed interval $[a;b]$. Then f is **continuous on** $[a;b]$ if f is continuous at all $x \in]a;b[$ and if f continuous on the right at a and on the left at b.*

Informally: a function is continuous on an interval if its curve can be drawn without lifting the pencil, or if the function is where you expect it to be if it is hidden by a vertical line.

Exercise 22

Determine whether $f : x \mapsto x^2$ is continuous on its domain.

Clearly, if f and g are continuous on an interval I then the sum, difference, product and quotient (if $g(x) \neq 0$) are continuous on I. Moreover, if g is continuous on an interval containing $f(I)$ then $g \circ f$ is continuous on I.

Exercise 23

Show, using the definition of continuity, whether the following functions are continuous on the given intervals.

(1) $f_1 : x \mapsto \frac{1}{3}x + \sqrt{2}$ on $\mathbb{R}$

(2) $f_2 : x \mapsto x^2 - 3x - 1$ on $\mathbb{R}$

(3) $f_3 : x \mapsto \dfrac{x+2}{x-1}$ on $]1; +\infty[$

Exercise 24

Determine whether $f : x \mapsto \frac{1}{x}$ is continuous on its domain.

Exercise 25

Prove that $x \mapsto \sqrt{x}$ is continuous on its domain i.e, for any value $x = a$ in the domain.

4

Derivative Functions

Definition 8
If a function is differentiable on a given interval I, then for any $x \in I$ the value $f'(x)$ exists. Hence we can define ***the derivative function*** *by*

$$f' : x \mapsto f'(x)$$

If $f'(a) = 0$, then in an ultrasmall neighbourhood of a the function is **stationary** – on an ultrasmall neighbourhood $[a - dx; a + dx]$ its variation is of the form $\varepsilon \cdot dx$ – its graph is indistinguishable from a horizontal line.

Exercise 26
Differentiate $f : x \mapsto x^2$ and $g : x \mapsto x^3$ at general x.

Notation: Let dx be ultrasmall relative to f and x. We write

$$\Delta f(a) = f(a + dx) - f(a) \text{ or } f(a + dx) = f(a) + \Delta f(a).$$

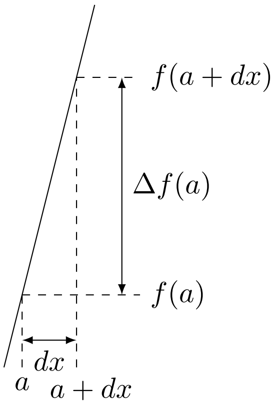
Hence, we have:

$$\frac{\Delta f(a)}{dx} \simeq f'(a).$$

Notation: A "$\simeq$" symbol may be replaced by a "$=$" symbol by adding a value ultraclose to zero on one of the sides i.e., $a \simeq b \Rightarrow a = b + \varepsilon$ where $\varepsilon \simeq 0$.

Hence

$$\frac{\Delta f(a)}{dx} = f'(a) + \varepsilon \text{ with } \varepsilon \simeq 0$$



Note: drawings involving ultrasmall or ultralarge values are not meant to be to scale nor be a correct representation. Their purpose is merely to help the mind.

Exercise 27

Prove the following theorem:

Theorem 7

If a real function f is differentiable at a then f is continuous at a.

(1) Give a direct proof.

(2) Give a proof by contrapositive.

Practice exercise 5 Answer page 21

Using definition 5, give the derivative functions of the following functions:

(1) $f : x \mapsto 3x + 2$

(2) $g : x \mapsto 2x^2 - x$

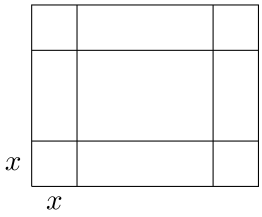
(3) $h : x \mapsto 5x^3 + 2x^2 - x$

(4) $k : x \mapsto 5x^3 + 2x^2 + 3x + 2$

In some cases, the slope to the right of a point is not the same as the slope to the left of that point. The derivative is the slope when it is the same on both sides.

Exercise 28

A factory wants to make cardboard boxes (with no top) out of sheets of $30cm \times 16cm$



The volume will be a function of x. The dimensions of the base are $30 - 2x$ and $16 - 2x$ (in centimetres). The height is x. What value(s) of x give(s) the maximum volume for the box?

4.1 Tangent line

Suppose f is differentiable at x_0. We observe that through a microscope, the curve of a function f at x_0 is indistinguishable from a straight segment. This straight segment meets the function at $\langle x_0; f(x_0)\rangle$ and there is a unique line which extends this segment with slope equal to the derivative which is indistinguishable from the curve. This line is the tangent line.

Definition 9
Let f be differentiable at x_0. The tangent line T_{x_0} is the unique line through $\langle x_0; f(x_0)\rangle$ with slope $f'(x_0)$.
It is the only straight line T which satisfies $T(x_0) = f(x_0)$ and $T'(x_0) = f'(x_0)$.

Exercise 29
Let $f : x \mapsto x^2$. Find the equation of the straight line tangent to f at $x = 3$.

Exercise 30
Show that

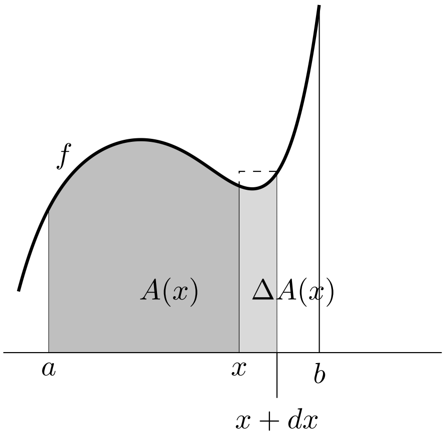
$$T_{x_0} : x \mapsto f'(x_0)(x - x_0) + f(x_0).$$

Exercise 31
Give the equation of the line tangent to $x \mapsto x^3 - 3 \cdot x^2$ at $x = 2$. For which values of x is this tangent horizontal?

4.2 Area under a curve

Consider a nonnegative function f continuous on a closed interval $[a; b]$. Note $A(x)$ the area between the curve of f and the horizontal x-axis.

The variation between x and $x + dx$ is $\Delta A(x)$.



Exercise 32

Using the drawing above, consider $f : x \mapsto 3x^2 + x$ between 2 and $2 + dx$.

(1) Write the formula for the variation of the area $\Delta A(2)$ or at least for upper and lower bounds to $\Delta A(2)$.

(2) Determine the equation of A.

Exercise 33

Calculate the area under $f : x \mapsto x^2$ and above the x-axis, between 2 and 5 i.e., $a = 2$ and $x = b$. Use that $A(2) = 0$

Answers to practice exercises

Answers to practice exercice 4, page 14

(1) 10

(2) -70

(3) 10

(4) 20

Answers to practice exercice 5, page 18

(1) $f'(x) = 3$

(2) $g'(x) = 4x - 1$

(3) $h'(x) = 15x^2 + 4x - 1$

(4) $k'(x) = 15x^2 + 4x + 3$

5
Differentiation Rules

Exercise 34

Prove the following theorem:

Theorem 8
Relative to an observation level: If a and b are observable and if $a \simeq x$ and $b \simeq y$,

(1) $a + b \simeq x + y$

(2) $a - b \simeq x - y$

(3) $a \cdot b \simeq x \cdot y$

(4) if also and $b \neq 0$, *then* $\dfrac{a}{b} \simeq \dfrac{x}{y}$.

(5) $a \simeq b \Rightarrow a = b$

Since observable numbers remain observable if we zoom further in, a property is not changed if the context is extended.

For the following rules, the proofs proceed by five steps:

(1) Definition of the derivative.

(2) Definition of the Δ.

(3) Definition of operations on functions.

(4) Expansion of $f(a + dx)$ as $f(a) + \Delta f(a)$.

(5) Division by dx.

(6) Algebra.

Exercise 35

Explain why if f is differentiable at a, then $\Delta f(a) \simeq 0$.

Exercise 36

Prove the following theorem:

Theorem 9

Let f and g be two real functions differentiable at a. Then the function $f \cdot g$ is differentiable at a and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

Exercise 37

Using the derivatives of $f : x \mapsto x^2$ and $g : x \mapsto x^3$, calculate the derivative of $h : x \mapsto x^5$ $(= x^2 \cdot x^3)$.

Exercise 38

Prove the following theorem:

Theorem 10

Let $c \in \mathbb{R}$ and $f : x \mapsto c$, for $x \in \mathbb{R}$

$$f'(x) = 0.$$

Exercise 39

Prove the following theorem:

Theorem 11

Let $c \in \mathbb{R}$ and f be a real function differentiable at a. Then the function $c \cdot f$ is differentiable at a and

$$(c \cdot f)'(a) = c \cdot f'(a).$$

The following theorem expresses a property for all natural numbers:

Theorem 12

$$(x^n)' = n \cdot x^{n-1}.$$

It is of course impossible to prove all cases. We prove by induction.

If

(1) The property holds for $n = 0$ (or $n = 1$),

(2) Assuming the property holds for n greater than 0 (or 1), we can prove that it also holds for $n + 1$,

then the property holds for all n.

The proof that this method of proof is valid is by contradiction. Assume (1) and (2) have been checked but that there is a value m such that the property does not hold for m. Let n be the smallest number such that the property does not hold. (This number is not zero because of (1).) Then the property holds for $n - 1$. But by (2), this proves that the property holds for n: a contradiction. So there can be no number for which the property does not hold.

Exercise 40

Try to prove theorem 12.

Exercise 41

Prove the following theorem:

Theorem 13

Let f and g be real functions differentiable at a. Then the function $f+g$ is differentiable at a and

$$(f+g)'(a) = f'(a) + g'(a).$$

Exercise 42

Find the derivatives of $h : x \mapsto x^3 + x^2$ and $k : x \mapsto 5x^3 - 7x^2$.

Exercise 43

Prove:

Theorem 14

Let f and g be two real functions differentiable at a and $g(a) \neq 0$. Then the function $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}.$$

Exercise 44

Find the slope of $x \mapsto \dfrac{x^2 - 2x}{x^3 + x^2}$ at $x = 1$.

Exercise 45

Find the derivative of

$$f : x \mapsto \frac{x}{x^2+1}$$

Exercise 46

Show that for $m \in \mathbb{Z}$

$$(x^m)' = m \cdot x^{m-1}.$$

Practice exercise 6 Answer page 31

Differentiate the following for general x:

(1) $f : x \mapsto 5x^4 + x^3 - 2x^2 + 25$

(2) $g : x \mapsto 5\sqrt{3}\, x^2 - 100$

(3) $h : x \mapsto \dfrac{x^2 + 2x - 1}{x^3 - 5}$

(4) $j : x \mapsto 5x^4 + \dfrac{1}{3x^2 - 2x + \pi}$

(5) $k : x \mapsto (5x + 2) \cdot \dfrac{1}{5x + 2}$

(6) $l : x \mapsto \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4}$

(7) $m : x \mapsto \dfrac{1 + x}{1 + \frac{1+x}{x^2}}$

Practice exercise 7 Answer page 31

Sketch the curve of $y = -(x - 3)(x + 1)(x - 1)$.

Practice exercise 8 Answer page 31

Let $y = \dfrac{10x}{x^2 + 1}$. Sketch the curve and give the equation of the line tangent to the curve at $x = 3$.

Practice exercise 9 Answer page 32

Consider each of the following as a function f, find the corresponding derivative function f'.

(1) $x^3 + x^2 + 2x - 4$

(2) $-x^3 + 2x^2 - 2x + 1$

(3) $\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$

(4) $\frac{1}{3}(x - 2)^3$

(5) $\dfrac{x^2}{x + 2}$

(6) $x - 1 + \dfrac{9}{x + 1}$

(7) $\dfrac{4x^2 + 4x + 5}{4x + 2}$

(8) $\dfrac{-x^2 - 2x - 1}{x + 3}$

(9) $|x - 2|$

(10) $\dfrac{x^2}{|x| + 2}$

(11) $x + 2 - \dfrac{1}{x + 1}$

(12) $|x^3 - 6x^2 + 11x - 6|$

Exercise 47

Find the derivative of the following functions. Since they are piecewise defined, the answer will be in 3 parts – one special point is the meeting point for bothe rules.

(1)

$$f : x \mapsto \begin{cases} x^2 & \text{if } x \geq 1 \\ 2x - 1 & \text{if } x < 1 \end{cases}$$

(2)

$$g : x \mapsto \begin{cases} x^2 & \text{if } x > 2 \\ x + 2 & \text{if } x \leq 2 \end{cases}$$

(3)

$$h : x \mapsto \begin{cases} x^2 & \text{if } x \geq 3 \\ 2x & \text{if } x < 3 \end{cases}$$

Exercise 48

Prove the following theorem, assuming that $\Delta g(a) \neq 0$:

Theorem 15 (Chain Rule)

Let f and g be real functions such that g is differentiable at a and f is differentiable at $g(a)$. The the function $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Exercise 49

Prove that the theorem holds also if $\Delta g(a) = 0$.

Exercise 50

Give the derivatives of the following functions:

(1) $f : x \mapsto (x^3 + 2x)^4$

(2) $g : x \mapsto (5x^3 + 3x^2)^{13}$

Exercise 51

Use $(\sqrt{x})^2 = x$ and theorem 15 to find the derivative of $y = \sqrt{x}$ (for $x > 0$) – assuming it exists.

Exercise 52

Give the derivatives of the following functions:

(1) $f : x \mapsto (\sqrt{x} + 1)^4$

(2) $g : x \mapsto \sqrt{5x^3 + 3x^2}$

(3) $h : x \mapsto \sqrt{x^2}$

Exercise 53

Find the derivatives of the following:

(1) $y = \sqrt{3x^3 + 2x + 1}$

(2) $y = (x^2 + 3)^5$

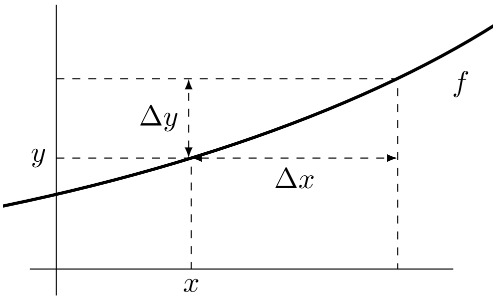
(3) $y = (ax + b)^n$

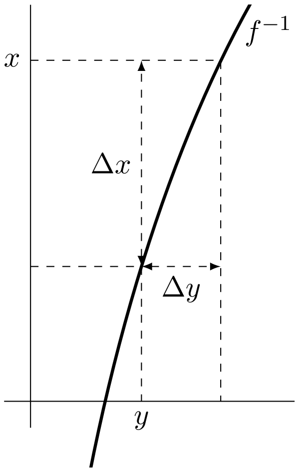
(4) $y = \sqrt{x^3 + 1}$

Let f be a function. Recall that the inverse function of f, if it exists, is written f^{-1} and is such that $f^{-1}(f(x)) = x$ amd if we write $f(x) = y$ then we also have $f(f^{-1}(y)) = y$.

⚠ $f^{-1}(x)$ is not $\dfrac{1}{f(x)}$.

A function has an inverse if the image of its curve by a symmetry through the $y = x$ axis is the curve of a function.





The slope of the tangent of the inverse is the reciprocal of the slope of the original tangent:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Theorem 16 (Derivative of the Inverse)
If $f : I \to J$ is a function, differentiable on I and has an inverse f^{-1}, and $f'(a) \neq 0$ then this inverse is differentiable at $b = f(a) \in J$ and

$$\frac{df^{-1}(b)}{dy} = \frac{1}{f'(a)}.$$

In general form:

$$\frac{df^{-1}(y)}{dy} = \frac{1}{f'(x)}$$

Exercise 54
Find the derivative of $y = x^{\frac{1}{n}}$.

This shows that the rule in exercise 12 holds also for rational n.

Exercise 55

Use $\mid x \mid = \sqrt{x^2}$ to find an expression for the derivative of $\mid x \mid$.

Theorem 17 (Derivative at a maximum or a minimum.)

Let f be a real function defined on an open interval $]a;b[$ differentiable at $c \in]a;b[$.
If $f(c)$ is a local maximum (or a local minimum) then $f'(c) = 0$.

Exercise 56

Prove theorem 17. (Hint, consider that the derivative must be ultraclose to $\dfrac{\Delta f(c)}{dx}$ whether dx is positive or negative.)

Optimisation and Other Problems

Exercise 57

A $1l$ milk pack is made of cardboard. Its sides can only be rectangles. The height is twice one of the other two dimensions. The area of the outside of the pack must be minimal.

What are the dimensions of the pack?

Exercise 58

Imagine you want to protect a part of a rectangular garden against a wall. You have 100m of fence. (No fence is needed against the wall.)

What is the biggest area that you can protect?

Exercise 59

A cylindrical jar has a volume defined by its radius and its height. If it contains one litre (1dm^3), what are the dimensions that will make it have the least outside area?

Exercise 60

Find the length and width of the rectangle inscribed within the ellipse given by the formula $4x^2 + y^2 = 16$ (sides parallel to the coordinate axes) such that its area is maximal.

Exercise 61

Let $\mathcal{P}$ be the parabola given by $x \mapsto x^2$ and A be the point $\langle 0; 5 \rangle$.

Find the point(s) on the parabola $\mathcal{P}$ such that its (their) distance to A is minimal.

Exercise 62

(1) Find the slope of the curve given by $y = 5x^3 - 25x^2$ at $x = 3.5$.

Equivalent statement: compute $f'(x)\Big|_{x=3.5}$

(2) Find the equation of the line tangent to the curve at $x = 1$.

Exercise 63

(1) For $f : x \mapsto x^2 + 5$ and the point $A\langle 0; 0\rangle$, what is the equation of the line passing through A, and tangent to f? Is it unique?

(2) If $g : x \mapsto ax^2 + b$, what values do a and b take to make $g(x)$ tangent to $t : x \mapsto 3x - 2$ at $x = 5$? What are the coordinates of the contact point?

A simplified writing can be used to remember differentiation rules: we have already used the writing $y = f(x)$ where y is a dependent variable and x the independent variable. When several functions are used, we can write $u = f(x)$ and $v = g(x)$, then we have (for constant c):

- $c' = 0$
- $(c \cdot u)' = c \cdot u'$
- $(u + v)' = u' + v'$
- $(u \cdot v)' = u' \cdot v + u \cdot v'$
- $\left(\dfrac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}$
- $(u \circ v)' = u' \cdot v'$ (in this case, u depends on v which depends on x).

Answers to practice exercises

Answers to practice exercice 6, page 25

(1) $f'(x) = 20x^3 + 3x^2 - 4x$

(2) $g'(x) = 10\sqrt{3}x$

(3) $h'(x) = -\dfrac{x^4 + 4x^3 - 3x^2 + 10x + 10}{(x^3 - 5)^2}$

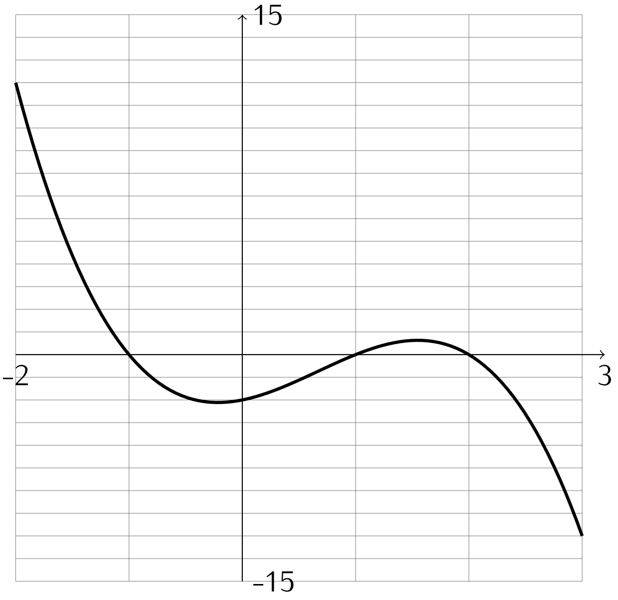
(4) $j'(x) = 20x^3 - \dfrac{6x - 2}{(3x^2 - 2x + \pi)^2}$

(5) $k'(x) = 0$

(6) $l'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} - \dfrac{3}{x^4} - \dfrac{4}{x^5}$

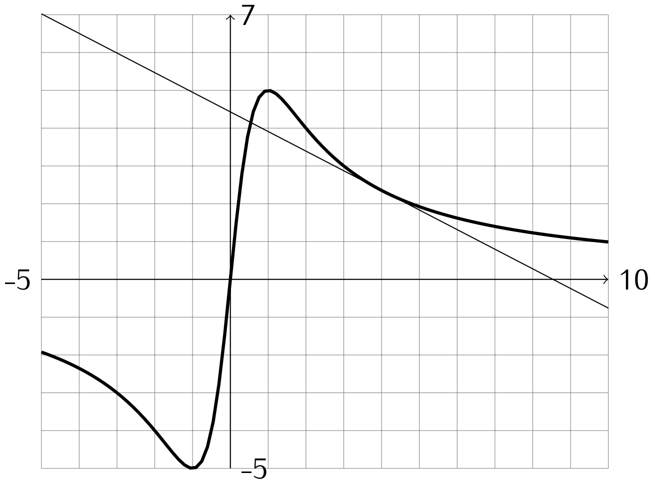
(7) $m'(x) = \dfrac{(x^2 + x + 1)(3x^2 + 2x) - (x^3 + x^2)(2x + 1)}{(x^2 + x + 1)^2} = \dfrac{x(x^3 + 2x^2 + 4x + 2)}{(x^2 + x + 1)^2}$

Answers to practice exercice 7, page 26



Answers to practice exercice 8, page 26

Tangent line is $y = -\dfrac{4}{5}x + \dfrac{27}{5}$



Answers to practice exercice 9, page 26

(1) $3x^2 + 2x + 2$

(2) $-3x^2 + 4x - 2$

(3) $x^2 - 5x + 6$

(4) $(x - 2)^2$

(5) $\dfrac{x(x + 4)}{(x + 2)^2}$

(6) $\dfrac{x^2 + 2x - 8}{(x + 1)^2}$

(7) $\dfrac{4x^2 + 4x - 3}{(2x + 1)^2}$

(8) $-\dfrac{x^2 + 6x + 5}{(x + 3)^2}$

(9) $\begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \\ \text{not differentiable} & \text{if } x = 2 \end{cases}$

(10) $\begin{cases} \dfrac{x(x + 4)}{(x + 2)^2} & \text{if } x \geq 0 \\ \\ \dfrac{-x(x - 4)}{x - 2)^2} & \text{if } x \leq 0 \end{cases}$

(11) $\dfrac{x^2 + 2x + 2}{(x + 1)^2}$

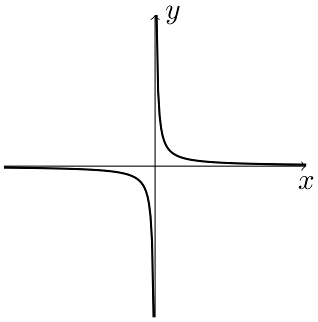
(12) $\begin{cases} 3x^2 - 12x + 11 & \text{if } x \in]1; 2[\cup]3; \infty[\\ -3x^2 + 12x - 11 & \text{if } x \in]-\infty; 1[\cup]2; 3[\\ \text{not differentiable} & \text{if } x \in \{1; 2; 3\} \end{cases}$

6

Asymptotes

Exercise 64

Consider the real function $f : x \mapsto \frac{1}{x}$.



(1) What is the domain of this function? Specify the context.

(2) What happens to the curve close to the vertical axis i.e., for values of x close to 0? Consider ultrasmall values of x.

(3) What happens to the curve close to the horizontal axis? i.e., for very large values of x? Consider ultralarge values of x (positive or negative).

(4) Draw this function for a horizontal range of $[-100; 100]$ and a vertical range of $[-100; 100]$.

Informally: For a given function f, a straight line is **an asymptote** of the function f if it is ultraclose to the function when either

- x is ultralarge (horizontal or oblique asymptote).
- y (or $f(x)$) is ultralarge (vertical asymptote).

Definition 10
A real function f ***has a vertical asymptote at*** *$x = a$ if $f(x)$ is positive or negative ultralarge for $x \simeq a$, x being less than a or x being greater than a.*
If it is the case for x greater than a, we write

$$x \simeq a_{+} \Rightarrow f(x) \text{ is ultralarge}$$

If it is the case for x less than a, we write

$$x \simeq a_{-} \Rightarrow f(x) \text{ is ultralarge}$$

Example: The function $f : x \mapsto 1/x$ has a vertical asymptote at 0. The only parameter of the function is 1, always observable. If dx is a positive ultrasmall number then $f(dx)$ is positive ultralarge. Hence

$$\frac{1}{dx} \text{ is ultralarge}$$

Definition 11
A real function f has a ***horizontal asymptote on the right*** *(resp. on the left) if there is an observable number L such that*

$$x \text{ ultralarge positive (resp. negative) } \Rightarrow f(x) \simeq L.$$

Example: Consider

$$\frac{x^2 - 3x + 1}{x^2 + 1} \text{ for ultralarge } x.$$

This means: consider the fraction for an ultralarge value of x.

The function $f : x \mapsto \dfrac{x^2 - 3x + 1}{x^2 + 1}$ is defined on $\mathbb{R}$. $1, 2$ and 3 are always observable. Let x be ultralarge. Then

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 + 1} = \frac{x^2(2 - \frac{3}{x} + \frac{1}{x^2})}{x^2(1 + \frac{1}{x^2})} = \frac{2 - \overbrace{\frac{3}{x}}^{\simeq 0} + \overbrace{\frac{1}{x^2}}^{\simeq 0}}{1 + \underbrace{\frac{1}{x^2}}_{\simeq 0}} \simeq \frac{2}{1} = 2,$$

hence f has a horizontal asymptote $y = 2$.

We now define the oblique asymptote

Definition 12
A real function f has an ***oblique asymptote at*** *on the right (resp. on the left) if there exist observable numbers a, b such that, if x is ultralarge positive (resp. negative), then*

$$f(x) - (ax + b) \simeq 0$$

The line $y = ax + b$ is the **oblique asymptote of** f

The existence of an oblique asymptote is a property of f hence the context is f.

This is equivalent to saying that $f(x) \simeq ax + b$ whenever x is ultralarge.

Example: Consider

$$f : x \mapsto \frac{x^3 + 2x^2 + x - 1}{x^2 + 1}$$

defined on $\mathbb{R}$. Using long division we have

$$f(x) = x + 2 - \frac{3}{x^2 + 1}.$$

Let x be ultralarge. We have

$$f(x) - (x + 2) = \frac{-3}{x^2 + 1} \simeq 0,$$

because $x^2 + 1$ is ultralarge. Hence f has an oblique asymptote at $y = x + 2$, i.e., $a = 1$ and $b = 2$.

Exercise 65

Find the asymptotes (if any) of

(1) $f : x \mapsto \dfrac{x}{2x^2 + 1}$

(2) $g : x \mapsto \dfrac{2x^2 + 1}{x}$

(3) $h : x \mapsto \dfrac{x^3 + 2}{2x^2 - 1}$

(4) $i : x \mapsto \dfrac{x^2 + 2x + 1}{x + 1}$

(5) $j : x \mapsto \dfrac{3x^3 + 2x^2 - x + 12}{x^2 + 8}$

For functions which are not rational functions, where the polynomial long division does not apply, we have the following:

Theorem 18

Let f be a real function and let a and b be observable (context is f). Then f has an oblique asymptote at $y = ax + b$ on the right (resp. on the left) if and only if, for ultralarge positive (resp. negative) x, there are observable a and b such that

$$\frac{f(x)}{x} \simeq a \quad \textit{and} \quad (f(x) - ax) \simeq b.$$

Remark: If $a = 0$ the line $y = ax + b$ becomes $y = b$ i.e., a horizontal asymptote.

Exercise 66

Prove the previous theorem.

Example: Consider $f : x \mapsto \sqrt{x^2+1}$ defined on $\mathbb{R}$. Let x be positive ultralarge. Then

$$\frac{f(x)}{x} = \frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2(1+1/x^2)}}{x} = \frac{|x|\overbrace{\sqrt{1+1/x^2}}^{\simeq 1}}{x} \simeq \begin{cases} 1 & \text{it } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Moreover:

$$f(x) - x = \sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1}-x)\cdot(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x} \simeq 0.$$

Hence f has an oblique asymptote at $y = x$ on theb right.

On the left, the function has an oblique asymptote at $y = -x$.

Exercise 67

Find the asymptotes at infinity (if any) of

(1) $i : x \mapsto x^{\frac{3}{2}}$

Practice exercise 10 Answer page 40

Find all asymptotes of the following functions.

(1) $f_1 : x \mapsto \dfrac{x^2-x}{x-1}$

(2) $f_2 : x \mapsto \dfrac{4x^3+2x^2-5}{3x^3-4x^2}$

(3) $f_3 : x \mapsto \sqrt{x^2+x}$

(4) $f_4 : x \mapsto \dfrac{\sqrt{x^5+x}}{\sqrt{3x^5-x}}$

(5) $f_7 : x \mapsto \dfrac{x^10}{x^10+1}$

Theorem 19 (Rule of de l'Hospital for $0/0$)

Let f and g be differentiable functions at a. Suppose that $f(a) = g(a) = 0$, but that $g'(a) \neq 0$. Then

$$\frac{f(a+dx)}{g(a+dx)} \simeq \frac{f'(a)}{g'(a)}$$

(provided $f'(a)$ and $g'(a)$ exist).

Exercise 68

Prove theorem 19.

The rule of de l'Hospital also holds for the case:

$$x \simeq a \Rightarrow \frac{f(x)}{g(x)} \simeq \frac{f'(x)}{g'(x)}$$

if $g'(x) \not\simeq 0$ and for the case $\frac{\text{ultralarge}}{\text{ultralarge}}$.

Exercise 69

Evaluate using de L'Hospital's rule.

(1) $\dfrac{1/t-1}{t^2-2t+1}$ for $t \simeq 1$ (with $(t>1)$).

(2) $\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ for $x \simeq 1$.

(3) $\dfrac{x^2}{\sqrt{2x+1}-1}$ for $x \simeq 0$.

(4) $\dfrac{2+1/t}{3-2/t}$ for $t \simeq 0$.

(5) $\dfrac{x+5-2x^{-1}-x^{-3}}{3x+12-x^{-2}}$ for ultralarge x

(6) $\left(t+\dfrac{1}{t}\right)((4-t)^{3/2}-8)$ for $t \simeq 0$.

(7) $\dfrac{u+u^{-1}}{1+\sqrt{1-u}}$ for ultralarge u.

Practice exercise 11 Answer page 40

Evaluate using de L'Hospital's rule.

(1) $\dfrac{\sqrt{9+x}-3}{x}$ for $x \simeq 0$

(2) $\dfrac{2-\sqrt{x+2}}{4-x^2}$ for $x \simeq 2$

(3) $\dfrac{\sqrt{u+1}+\sqrt{u-1}}{u}$ for ultralarge u

(4) $\dfrac{(1-x)^{1/4}-1}{x}$ for $x \simeq 0$

(5) $\left(\dfrac{1}{t}+\dfrac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$ for $x \simeq 0_+$

(6) $\dfrac{(u-1)^3}{u^{-1}-u^2+3u-3}$ for $u \simeq 1$

(7) $\dfrac{1+5/\sqrt{u}}{2+1/\sqrt{u}}$ for $u \simeq 0_+$

(8) $\dfrac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$ for ultralarge x

(9) $\dfrac{1-t/(t-1)}{1-\sqrt{t/(t-1)}}$ for ultralarge t

7

Curve Sketching

Curve sketching needs the following steps:

- Find the domain.
- Find the roots and the intercept (if any).
- Find the asymptotes (if any).
- Find the derivative (if any).
- Find the roots of the derivative (if any).
- Determine the maximums and minimums.
- Put all these values in a table.
- Draw arrows which indicate the general direction of the function:
- Use this information to choose a convenient scale.
- Sketch the function.

Practice exercise 12 Answer page 40

(1) $f_1 : x \mapsto \dfrac{x^2}{x+2}$

(2) $f_2 : x \mapsto x - 1 + \dfrac{9}{x+1}$

(3) $f_3 : x \mapsto \dfrac{-x^2 - 2x - 1}{x+3}$

(4) $f_4 : x \mapsto x + 3 + \dfrac{1}{2x+1}$

(5) $f_5 : x \mapsto \dfrac{x^2 - 4x + 6}{(x-2)^2}$

(6) $f_6 : x \mapsto \dfrac{2x^2 - 3}{x^2 - 1}$

(7) $f_7 : x \mapsto \dfrac{x^2 + 3x - 4}{x^2 - x - 2}$

(8) $f_8 : x \mapsto \dfrac{x^3 + 2}{2x}$

(9) $f_9 : x \mapsto \dfrac{x^3 - 1}{x^2}$

(10) $f_{10} : x \mapsto \dfrac{2x - 1}{\sqrt{x^2 + 2}}$

(11) $f_{11} : x \mapsto \dfrac{\sqrt{x^2 + 1}}{x + 1}$

(12) $f_{12} : x \mapsto \dfrac{\sqrt{x^2 - 4x + 3}}{x + 1}$

Answers to practice exercises

Answers to practice exercice 10, page 36

Vertical asymptote of the form $x = c$, horizontal asymptote of the form $y = b$, oblique asymptote of the form $y = ax + b$.

(1) $y = x$

(2) $y = 1$, $x = 0$, $x = 4/3$

(3) $\begin{cases} y = x & \text{if } x > 0 \\ y = -x & \text{if } x < 0 \end{cases}$

(4) $y = \sqrt{1/3}$, $x = \sqrt[4]{1/3}$

(5) $\begin{cases} y = 0 & \text{if } x < 0 \\ y = 1 & \text{if } x > 0 \end{cases}$

Answers to practice exercice 11, page 37

(1) $1/6$

(2) $1/16$

(3) 0

(4) $-1/4$

(5) $1/2$

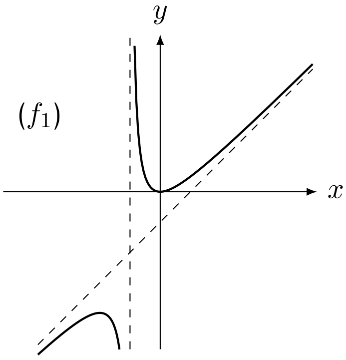
(6) -1

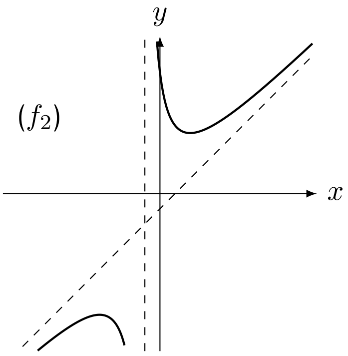
(7) 5

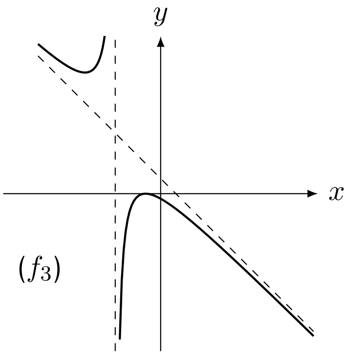
(8) ultralarge

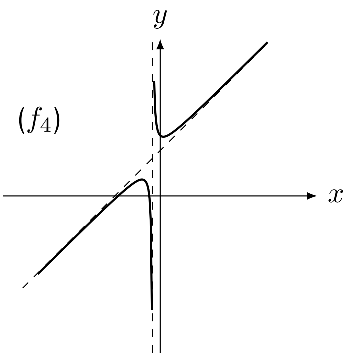
(9) 2

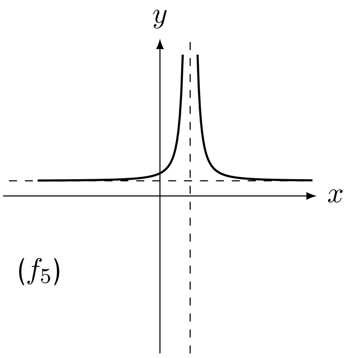
Answers to practice exercice 12, page 39

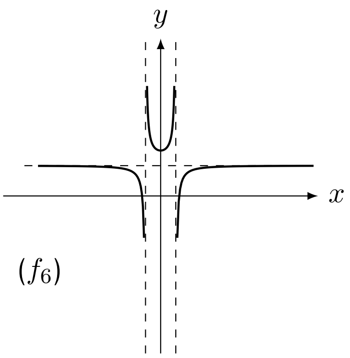












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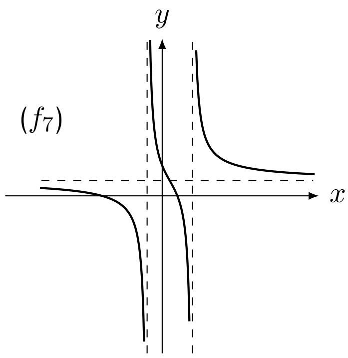

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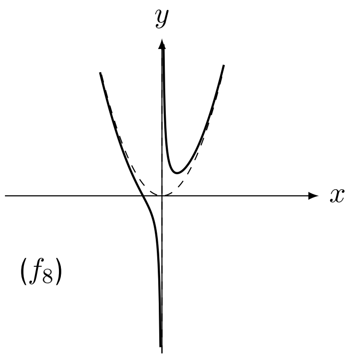

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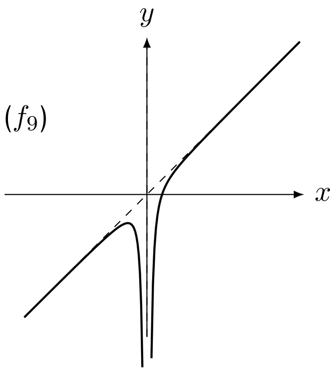

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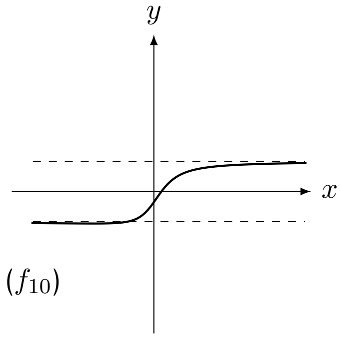

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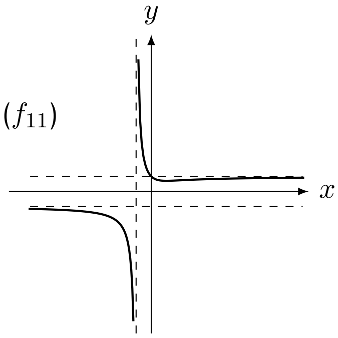

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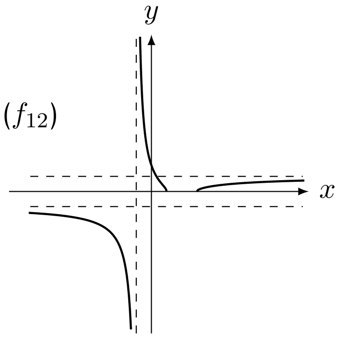


Analysis

with ultrasmall numbers

Standard Level

PART II

8
Continuity and Differentiability

Theorem 20 (Intermediate Value theorem)
Let f be a real function continuous on $[a;b]$. Let d be a real number between $f(a)$ and $f(b)$. Then there exists c in $[a;b]$ such that $f(c) = d$.

This theorem does not tell us how to find the root or the value c such that $f(c) = d$. It only asserts the *existence* of such a number. For specific functions where we can calculate explicitly the roots this theorem is not really necessary but, when proving theorems about continuous functions in general, it is the only way to know that there is a root.

Exercise 70
Give an example of a function f discontinuous on $[a;b]$ with $f(a) < 0$ and $f(b) > 0$ such that there is no c in the interval $[a;b]$ such that $f(c) = 0$.

Definition 13
*A function is **smooth** if it is differentiable and its derivative is continuous.*

Allmost all functions encountered so far are smooth.

Definition 14
*A function has **maximum** (respectively **minimum**) on an interval I if there is a $c \in I$ such that for any $x \in I$ we have $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$).*
*If a point is either a maximum or a minimum, it is an **extremum**.*

Theorem 21 (Extreme value)
Let f be a function continuous on $[a;b]$ smooth on $]a,b[$. Then it has a (local) maximum and a (local) minimum on $[a;b]$.

Theorem 22 (Rolle)
Let f be a real function continuous on $[a;b]$ and smooth on $]a;b[$. If $f(a) = f(b)$, then there is a $c \in]a;b[$ such that

$$f'(c) = 0.$$

Exercise 71

Prove Rolle's theorem.

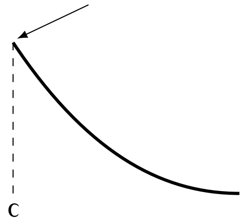
Theorem 23 (Critical Point Theorem)

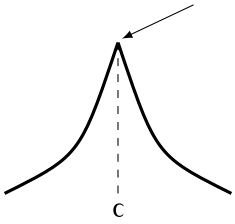
Let f be a function smooth on I and suppose that c is a point in I and f has either a maximum or a minimum at c. Then one of the following three things must happen:

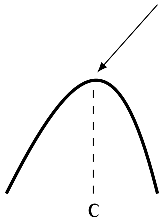
(1) c is an end point of I.

(2) $f'(c)$ is undefined.

(3) $f'(c) = 0$





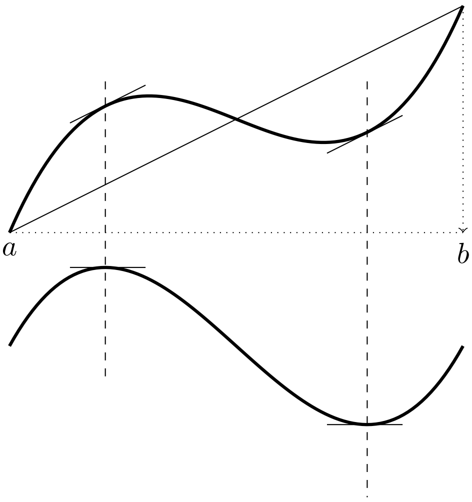


Theorem 24 (Mean Value)

Let f be a real function continuous on $[a;b]$ and smooth on $]a;b[$. Then there is a $c \in]a;b[$ such that

$$f(b) - f(a) = f'(c) \cdot (b - a).$$

Consider g which is obtained by subtracting the line $\ell(x)$ through $(a, f(a))$ and $(b, f(b))$ from the function f i.e., $g(x) = f(x) - \ell(x)$.



Exercise 72

Show that g satisfies Rolle's theorem and conclude with the mean value theorem.

Variation

We now make the link between global variation and derivative.

Definition 15
Let f be a real function defined on an interval I.

(1) *The function f is* ***increasing on*** *I if $f(x) \leq f(y)$, whenever $x < y$ in I.*

(2) *The function f is* ***decreasing on*** *I if $f(x) \geq f(y)$, whenever $x < y$ in I.*

If the inequalities are strict, then we say that the function is strictly increasing or strictly decreasing.

Theorem 25 (Variation and Derivative)
Let f be a real function differentiable on an interval I. Then

(1) *If $f'(x) \geq 0$ (> 0) whenever $x \in I$ then f is (resp. strictly) increasing on I.*

(2) *If $f'(x) \leq 0$ (< 0) whenever $x \in I$ then f is (resp. strictly) decreasing on I.*

(3) *If $f'(x) = 0$ whenever $x \in I$ then f is constant on I.*

The converse is obvious: if f is increasing at a, then $f'(a) > 0$.

Exercise 73
Prove theorem 25 using the mean value theorem.

Exercise 74
Prove the following theorem:

Theorem 26 (Uniqueness, up to an Additive Constant)
Let f and g be functions and I an interval.

$$f' = g' \iff \text{there is a real number } C \text{ such that } f = g + C$$

Theorem 28 is one direction of theorem 26. You will need theorem 25 (page 47) for the other direction.

9

Integrals

9.1 Area under a positive curve

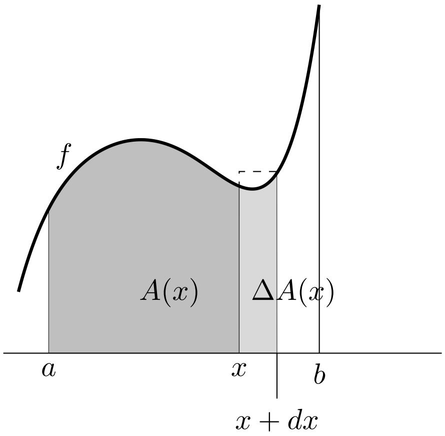
Theorem 27
Let f be a non-negative function continuous on $[a;b]$ and smooth on $]a,b[$. Then the function

$$A : x \mapsto A(x),$$

where $A(x)$ is the area under the curve of f between a and x has the following properties

(1) $A'(x) = f(x)$*, whenever* $x \in [a;b]$.

(2) $A(a) = 0$.



Exercise 75
Prove theorem 27.

Hint: Context is a, b, f and x. Let dx be ultrasmall. As f is smooth on $[x; x + dx]$ the function f reaches its maximum, say $(x_M, f(x_M))$, and its minimum, say $(x_m, f(x_m))$ in that interval (theorem 21).

Show that $\Delta A(x)$ is bounded above and below, that these bounds are ultraclose, then conclude.

Notation

$$A(b) - A(a) \text{ is written } A(x)\Big|_a^b$$

9.2 Antiderivative

Definition 16 (Antiderivative)
An antiderivative of a function f is a function A such that $A'(x) = f(x)$.

Newton assumed that gravitation is a constant acceleration. Given such an acceleration, how can one find the equation of position with respect to time?

What is the area under a curve and what is the relation between measuring areas and retrieving the function of position when velocity is known?

Exercise 76
Prove the following theorem:

Theorem 28
If A is an antiderivative of f, then for any constant $C \in \mathbb{R}$, $A + C$ is also an antiderivative of f.

Exercise 77
Find the antiderivatives for the following:

(1) $x \mapsto 3x$

(2) $x \mapsto x^2$

(3) $x \mapsto 5$

(4) $t \mapsto 3t + 5$

(5) $u \mapsto u^2 + 3u + 5$

(6) $v \mapsto v^3$

(7) $x \mapsto \frac{1}{\sqrt{x}}$

Check your results by differentiating them.

Exercise 78
Using $A' = f$ and $A(a) = 0$:

(1) Calculate the area between the curve and the x-axis for $y = x^2$ from $x = -5$ to $x = 5$.

(2) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = 0$ to $x = 3$.

(3) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -2$ to $x = 0$.

(4) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -10$ to $x = 10$.

Exercise 79

Calculate the area between $y = 5x^4 - 3x^3 + 2x^2 - 10$ and the x-axis from $x = -1$ to $x = 1$.

9.3 A $\int$um of $\int$lices

Exercise 80

Let $g : x \mapsto x^2$, $a = 0$ and $b = 5$.

(1) Cut the interval $[a; b]$ into an ultralarge number N of pieces. Put all these pieces together again – add all their lengths. What is the result?

Write this using the symbol for a sum i.e., sum for $k = 0$ to $N - 1$.

(2) For each $dx = \frac{b-a}{N}$ there is a corresponding Δy. Add all the Δy between $f(a)$ and $f(b)$. Find the result.

(3) Use the microscope equation to express Δy in terms of y or y'. Add all these terms. Find the result.

The (vertical) variation of f between a and b is written $f(x)\Big|_a^b$

Fundamental Theorem of Calculus

Definition 17

*Let f be a real function defined on $[a; b]$. Let n be a positive integer. Let $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \ldots, n$. We say that f **is integrable on** $[a; b]$ if there is an observable I such that for any ultralarge integer n with $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \ldots, n$, we have*

$$\sum_{i=0}^{n-1} f(x_i) \cdot dx \simeq I.$$

If such an I exists, it is called the integral of f between a and b; written

$$\int_a^b f(x) \cdot dx.$$

Note that this sum is defined whether f is positive or not.

Theorem 29

Let f be an function continuous on $[a; b]$ and smooth on $]a; b[$. Let $\frac{1}{N} \simeq 0$, $dx = \frac{b-a}{N}$ and $x_k = a + k \cdot dx$, then there exists a point $c \in [a; b]$ such that

$$f(c) \cdot (b - a) = \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

Exercise 81

Prove theorem 29. To prove this, use that since f is continuous on $[a;b]$ there is a minimum $f(m)$ and a maximum $f(M)$ of f on $[a;b]$. Replace all $f(x_k)$ one by $f(m)$ then by $f(M)$ to conclude that the sum is not ultralarge and then use continuity again to conclude the proof.

The $\int$ symbol is an elongated S and stands for the latin word "summa": a sum, since it can also be shown that instead of finding the area as a variation, it is a $\int$um of $\int$lices.

If f represents the velocity of an object, an object comming closer to the origin *reduces* its distance. An object which moved away and came back where it was has a final distance of zero. Hence we define

Theorem 30 (Fundamental theorem of Calculus)
Let f be a function continuous on $[a;b]$ and smooth on $]a;b[$. Let F be an antiderivative of f on $[a;b]$. Then

$$\int_a^b f(x) \cdot dx = F(b) - F(a).$$

Notation: we write

$$F(x)\Big|_a^b = F(b) - F(a).$$

If bounds are given, the integral represents a value: it is a **definite integral**. If no bounds are given, it represents an antiderivative: it is an **indefinite integral**.

Exercise 82

Let $f : x \mapsto x^2$,

Calculate the area under $f'(x)$ between $x = 0$ and $x = 5$.

Exercise 83

Show that for a definite integral, it does not matter which antiderivative is chosen.

Exercise 84

What conditions would a function need to satisfy in order to be non-integrable? Give such a function.

Exercise 85

A constant function $f : x \mapsto C$ from a to b defines a rectangle. Check that the area under f is the “usual” formula: $(b - a) \cdot C$

Exercise 86

The function $y = x$ defines a triangle. Show that the area of the triangle from 0 to a yields the “usual” result for the area of a triangle.

Exercise 87

Sketch the curve of $f : x \mapsto x^2$ and $g : x \mapsto x^3$. Calculate the points where $f(x) = g(x)$ Calculate the geometric area of the closed surface between the two curves.

Integration Rules

Theorem 31 (Additivity of the integral)
Let f be a real function continuous on $[a;c]$ and $b \in [a;c]$. Then

$$\int_a^b f(x) \cdot dx + \int_b^c f(x) \cdot dx = \int_a^c f(x) \cdot dx.$$

Theorem 32 (Linearity of the integral)
Let f and g be real functions integrable on $[a;b]$. Let λ be a real number. Then

(1)

$$\int_a^b (\lambda \cdot f(x)) \cdot dx = \lambda \cdot \int_a^b f(x) \cdot dx$$

(2)

$$\int_a^b (f(x) + g(x)) \cdot dx = \int_a^b f(x) \cdot dx + \int_a^b g(x) \cdot dx.$$

Note that if f and g are integrable then all linear combinations of f and g are integrable.

Exercise 88

Prove theorem 32.

Exercise 89

For each of the following functions, find the general form of the antiderivative:

(1) $f : x \mapsto 8\sqrt{x}$

(2) $f : t \mapsto 3t^2 + 1$

(3) $f : t \mapsto 4 - 3t^3$

(4) $f : s \mapsto 7s^{-3}$

(5) $f : x \mapsto (x - 6)^2$

(6) $f : y \mapsto y^{\frac{3}{2}}$

(7) $f : x \mapsto |x|$

(8) $f : u \mapsto u^2 + u^{-2}$

(9) $f : x \mapsto 4$

(10) $f : t \mapsto t$

(11) $f : z \mapsto \dfrac{2}{z^2}$

Check your results by differentiating them.

Exercise 90

(1) If $F'(x) = x + x^2$ for all x, find $F(1) - F(-1)$.

(2) If $F'(x) = x^4$ for all x, find $F(2) - F(1)$.

(3) If $F'(t) = t^{\frac{1}{3}}$ for all t, find $F(8) - F(10)$.

Exercise 91

The following computation may seem correct: $\int_{-1}^{1} x^{-2}dx = -\frac{1}{x}\Big|_{-1}^{1} = -2$ yet there is no $x \in [-1, 1]$ such that $f(x) < 0$. Why is this not so?

Exercise 92

In the following problems an object moves along the y axis. Its velocity varies with respect to the time. Find how far the object moves between the given times t_0 and t_1.

(1) $v = 2t + 5$ $t_0 = 0$ $t_1 = 2$

(2) $v = 4 - t$ $t_0 = 1$ $t_1 = 4$

(3) $v = 3$ $t_0 = 2$ $t_1 = 6$

(4) $v = 3t^2$ $t_0 = 1$ $t_1 = 3$

(5) $v = 10t^{-2}$ $t_0 = 1$ $t_1 = 100$

Theorem 33 (Integration with inside derivative)
Let f and g be real functions differentiable on $[a; b]$ such that f' and g' are continuous on $[a; b]$. Then

$$\int_a^b f'(g(x)) \cdot g'(x) \cdot dx = f(g(x))\Big|_a^b.$$

Exercise 93

Prove theorem 33.

Variable substitution

Consider $\displaystyle\int_a^b f(x) \cdot dx$.

If x is a function of u written $x = g(u)$ then $dx = g'(u) \cdot du$,

$f(x)$ becomes $f(g(u))$ and the bounds must be changed to a_1 and b_1 so that $g(a_1) = a$ and $g(b_1) = b$

Example: Let

$$\int_0^1 \sqrt{1 + \sqrt{x}} \cdot dx.$$

Consider the variable change $u = 1 + \sqrt{x}$. Then $x = (u-1)^2 = g(u)$, the derivative of g is continuous. If $x = 0$ then $u = 1$ and if $x = 1$ then $u = 2$. Moreover $f(g(u)) = \sqrt{u}$ and

$$dx = 2 \cdot (u-1) \cdot du.$$

Replacing all terms we obtain

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx = 2\int_1^2 \sqrt{u} \cdot (u-1) \cdot du = 2\int_1^2 \left(u^{3/2} - u^{1/2}\right) \cdot du$$

so that

$$2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Bigg|_1^2 = \frac{8+8\sqrt{2}}{15}.$$

As g has an inverse which is $x \mapsto 1 + \sqrt{x}$ and is differentiable (except at $x = 0$), we can revert to the variable x and find an antiderivative:

$$\int \sqrt{1+\sqrt{x}} \cdot dx = \frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5 - \frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3 + C.$$

Exercise 94

Calculate

$$\int_0^1 \sqrt{5x+2} \cdot dx.$$

Use $u = 5x + 2$. Calculate du, change the bounds, calculate the integral.

Same integral. Use $v = \sqrt{5x+2}$

The difficulty is usually to find which variable substitution is best.

Exercise 95

Use variable substitution to evaluate the following:

(1) $\displaystyle\int_0^{10} \frac{1}{(2x+2)^2} \cdot dx$

(2) $\displaystyle\int (3-4z)^6 \cdot dz$

(3) $\displaystyle\int_{-1}^1 2t\sqrt{1-t^2} \cdot dt$

(4) $\displaystyle\int_a^b \sqrt{3y+1} \cdot dy$

(5) $\displaystyle\int \frac{4y}{(2+3y^2)^2} \cdot dy$

(6) $\displaystyle\int_{-2}^2 x(4-5x^2)^2 \cdot dx$

(7) $\displaystyle\int (1-x)^{\frac{3}{2}} \cdot dx$

Practice exercise 13 Answer page 68

(1) $\displaystyle\int_0^1 \frac{u}{\sqrt{1-u^2}} \cdot du$

(2) $\displaystyle\int_1^2 \frac{u}{\sqrt{1-u^2}} \cdot du$

(3) $\displaystyle\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx$

(4) $\displaystyle\int_0^{10} t(t^2+3)^{-2} \cdot dt$

(5) $\displaystyle\int_{\sqrt{6}}^5 x(x^2+2)^{\frac{1}{3}} \cdot dx$

(6) $\displaystyle\int_{-1}^1 \frac{x^2}{(4-x^3)^2} \cdot dx$

(7) $\displaystyle\int_1^2 \frac{1}{t^2\sqrt{1+\frac{1}{t}}} \cdot dt$

Antiderivative of $x \mapsto \dfrac{1}{x}$

Let n be a positive integer. From $(x^{n+1})' = (n+1) \cdot x^n$ we can deduce

$$\int x^n \cdot dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1.$$

Hence an antiderivative of $x \mapsto \dfrac{1}{x}$ is not a particular case of this formula.

Exercise 96

Let f be an antiderivative of $x \mapsto \frac{1}{x}$ (why is there one?). Then f is strictly increasing (why?) and so it has an inverse, call it g. Show that this implies $g'(x) = g(x)$.

Exercise 97

Let $a, b > 0$. Use the substitution $u = \frac{t}{a}$ to show that (considering f to be the antiderivative of $\frac{1}{x}$.)

$$\int_a^{a \cdot b} \frac{1}{t} \cdot dt = \int_1^b \frac{1}{u} \cdot du.$$

Deduce that $f(a \cdot b) = f(a) + f(b)$.

Exercise 98

Let $a > 0$ and b a rational number. Show that (considering f to be the antiderivative of $\frac{1}{x}$.)

$$f(a^b) = b \cdot f(a).$$

(To find the substition, consider the transformation of the bounds.)

Exercise 99

What kind of function has the properties $f(a \cdot b) = f(a) + f(b)$ and $f(a^b) = b \cdot f(a)$?

Theorem 34

The antiderivative f of $\frac{1}{x}$ satisfies the following properties

- $x \simeq 0_+ \Rightarrow f(x)$ *is ultralarge and negative*
- x *is ultralarge positive* $\Rightarrow f(x)$ *is ultralarge positive.*

Exercise 100

Prove theorem 34. Hint: for ultralarge x use ultralarge N such that $2^N \leq x$.

Definition 18

*The **natural logarithm** is the function* $\ln :]0; +\infty[\to \mathbb{R}$ *defined by*

$$x \mapsto \int_1^x \frac{1}{t} \cdot dt.$$

Definition 19

We define e to be the unique number such that

$$\ln(e) = 1.$$

e is an irrational number whose first digits are

$$e = 2.71828\ldots$$

Definition 20

*The **exponential** function* $\exp : \mathbb{R} \longrightarrow]0; +\infty[$ *is defined as the inverse of ln.*

We have, for rational x, that $a^x = \exp(x \ln(a))$, hence $e^x = \exp(x)$. For irrational x, we **define** a^x to be $\exp(x \ln(a))$ hence also $e^x = \exp(x)$ for all x.

We also have $\ln(a^y) = y \cdot \ln(a)$ for all y. Writing $x = a^y$ we get $\ln(x) = \log_a(x) \cdot \ln(a)$ so $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

The following property makes it a remarkable function.

Theorem 35

$$(\exp(x))' = \exp(x).$$

(this was proven by exercise 96).

Exercise 101

Let f be a positive real function whose derivative is continuous. Calculate:

$$\int \frac{f'(x)}{f(x)} \cdot dx$$

Exercise 102

Let f be a positive real function whose derivative is continuous. Calculate:

$$\int f'(x) \cdot e^{f(x)} \cdot dx$$

Exercise 103

(1) Differentiate $\ln(x)$.

(2) Differentiate e^x.

(3) Integrate $x \mapsto e^x$.

(4) Differentiate the function $x \mapsto \ln(\ln(x))$.

(5) Differentiate the function $x \mapsto \ln(x^a)$ (Note that a is not the variable!)

(6) Differentiate the function $x \mapsto \ln(a^x)$.

(7) Differentiate $x \mapsto e^{x^2}$.

(8) Using the fact that $u = e^{ln(u)}$ (if $u > 0$) differentiate $x \mapsto a^x$ (for $a > 0$ and $x > 0$).

(9) Same idea: Differentiate the function $x \mapsto x^x$.

Exercise 104

Differentiate $\ln(|x|)$.

This proves the following extension:

Theorem 36

The antiderivative of $\frac{1}{x}$ *is* $\ln(|x|) + K$ *for some constant* K.

Practice exercise 14 Answer page 68

Find the antiderivatives of the following functions:

- $f_a : x \mapsto 5x^4 - 2x + 4$
- $f_b : x \mapsto x^3 - 5x^2 + 3x - 2$
- $f_c : x \mapsto 2x - 1$
- $f_d : x \mapsto \frac{5}{4}x^4 - \frac{3}{4}x^2 + \frac{5}{2}x + \frac{3}{2}$
- $f_e : x \mapsto 2x + 1 - \frac{1}{x^2}$
- $f_f : x \mapsto 3 + \frac{2}{x^2} - \frac{5}{x^3}$
- $f_g : x \mapsto x^3 + \frac{1}{x^2}$
- $f_h : x \mapsto \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
- $f_i : x \mapsto \frac{1}{\sqrt{x}} + \sqrt{x}$

- $f_j : x \mapsto (x+1)^2$
- $f_k : x \mapsto 15(3x-2)^4$
- $f_l : x \mapsto (2x+1)^3$
- $f_m : x \mapsto (3-x)^{11}$
- $f_n : x \mapsto (3-4x)^4$
- $f_o : x \mapsto \sqrt{3x-2}$
- $f_p : x \mapsto \dfrac{1}{\sqrt{x-1}}$
- $f_q : x \mapsto 4x(3-x^2)^5$
- $f_r : x \mapsto (2x-3)(x^2-3x+1)^4$
- $f_s : x \mapsto (3x^2-4x+1)(x^3-2x^2+x+3)^2$
- $f_t : x \mapsto (4x^2-5x)^2(16x-10)$
- $f_u : x \mapsto (3x-1)(3x^2-2x+5)^3$
- $f_v : x \mapsto \dfrac{2x}{(x^2+1)^2}$
- $f_w : x \mapsto \dfrac{2x+1}{(x^2+x+3)^2}$
- $f_x : x \mapsto x\sqrt{x^2+1}$
- $f_y : x \mapsto \dfrac{3x^2}{\sqrt{9+x^3}}$
- $f_z : x \mapsto (3x^2+1)\sqrt{x^3+x+2}$
- $f_A : x \mapsto e^{2x}$
- $f_B : x \mapsto \dfrac{1}{e^{3x}}$
- $f_C : x \mapsto xe^{-x^2}$
- $f_D : x \mapsto 2^{-x}$
- $f_E : x \mapsto e^{2x}\sqrt{1+e^{2x}}$
- $f_F : x \mapsto x^2e^x$
- $f_I : x \mapsto \dfrac{1}{2x+3}$
- $f_J : x \mapsto \dfrac{2x}{x-1}$
- $f_K : x \mapsto \dfrac{x-1}{x+1}$
- $f_L : x \mapsto (\ln(x))^2$
- $f_N : x \mapsto \ln(x)$
- $f_O : x \mapsto \dfrac{x}{x+1}$
- $f_P : x \mapsto \dfrac{1}{x\ln(x)}$

Applications of the Integral

Mean value of a function

The mean value is unambiguous when we consider n points, where n is a positive integer. We now show that defining the mean value of a continuous function on $[a;b]$ as

$$\frac{1}{b-a}\int_a^b f(x)\cdot dx$$

is a natural extension of this concept.

Consider a continuous function f and the interval $[a;b]$. Context is a, b and f. Let N be a positive ultralarge integer. Let $dx = (b-a)/N$ and $x_i = a + i\cdot dx$, for $i = 1, \ldots, N$. Then the mean value of the function can be approximated by the mean value of the N points $f(x_i)$, $i = 0, \ldots, N-1$. But

$$\frac{\sum_{i=0}^{N-1} f(x_i)}{N} = \frac{dx}{b-a}\sum_{i=0}^{N-1} f(x_i) = \frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i)\cdot dx \simeq \frac{1}{b-a}\int_a^b f(x)\cdot dx,$$

since f is continuous on $[a;b]$.

The mean is the part of this number which is observable i.e., the integral. We therefore define:

Definition 21
*The **mean value** of a function f continuous on $[a;b]$ is*

$$\frac{1}{b-a}\int_a^b f(x)\cdot dx.$$

The mean value is a number μ such that the area under the curve is equal to $\mu\cdot(b-a)$, i.e., the height of a rectangle of basis $(b-a)$ whose (oriented) area is equal to the integral.

Theorem 37
If f is a function continuous on $[a;b]$, then there exists a point $c \in [a;b]$ such that $f(c)$ is the mean value of the function on $[a;b]$.

Note that theorem 37 is a restatement of the mean value theorem, for the antiderivative of f. When we claim that there is a $c \in [a;b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\cdot dx,$$

we are in fact asserting that there is a $c \in [a;b]$ such that

$$f(c)\cdot(b-a) = \int_a^b f(x)\cdot dx = F(b) - F(a),$$

and as $F'(x) = f(x)$, we conclude that there is a $c \in [a;b]$ such that $F'(c)\cdot(b-a) = F(b)-F(a)$.

Exercise 105

Calculate the mean value of $x \mapsto x^2$ on $[-4; 4]$.

Exercise 106

Calculate the mean value of $x \mapsto x^3$ on $[-4; 4]$.

Exercise 107

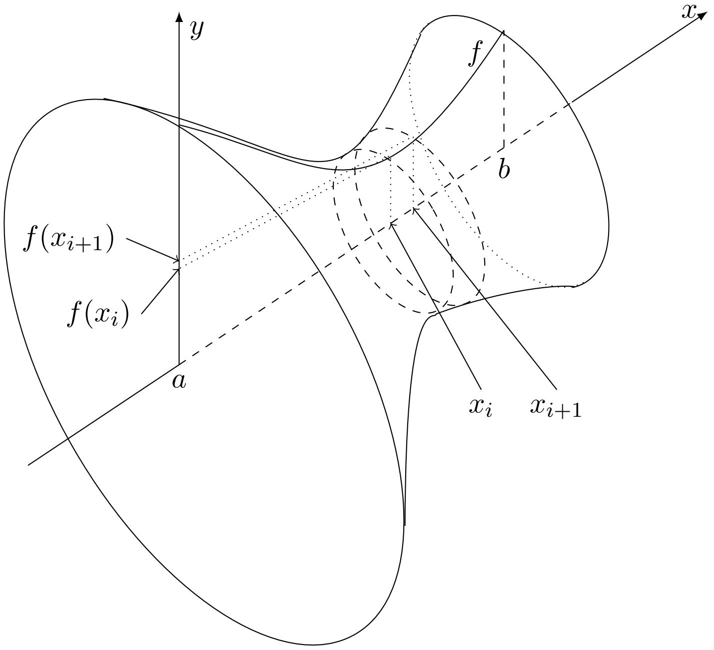
Let $f : x \mapsto x^2$ and the interval $[0; t]$. Find the value of t such that the mean value of f over the interval is equal to π.

Exercise 108

An object falling on earth satisfies the equation $d(t) = \frac{1}{2}gt^2$ where $g \approx 9.81[m/s^2]$, t is the time in seconds and $d(t)$ is the vertical distance.

If an object falls for $10s$, what is its average distance from its initial point?

Solid of Revolution



Exercise 109

An area is calculated by approximating the surface by ultrasmall rectangles. To find the formula for the volume of a solid of revolution, proceed in the same manner: consider that the solid is ultraclose to an ultralarge number of ultrathin disks. Find the formula for the volume of a solid of revolution given by a function f.

Exercise 110

Evaluate the volume of the solid of revolution of $y = \dfrac{1}{x}$ around the x-axis between $x = 1$ and $x = 10$.

Arc length

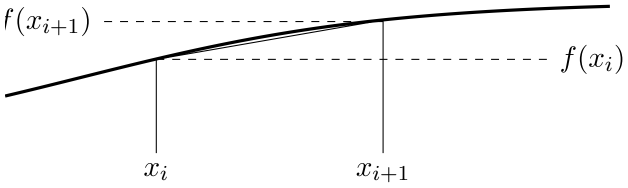
Exercise 111

Approximating the length of a curve by ultrasmall straight lines leads to the following definition. Explain why it is a reasonable definition (using the drawing).

Definition 22

Let $f : [a; b] \to \mathbb{R}$ be smooth. Then the graph of f has length

$$L = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx.$$



Exercise 112

Find the lengths of the following curves:

(1) $y = 2x^{3/2} \quad 0 \leq x \leq 1$

(2) $y = \dfrac{2}{3}(x+2)^{\frac{3}{2}} \quad 0 \leq x \leq 3$

10

Limits

If we want to study the behaviour of f in the neighbourhood of a, the function f must be defined *around* a, but not necessarily at a. If the function is defined in a neighbourhood of a, by closure, it is possible to use a neighbourhood defined by observable bounds. Hence $f(x)$ must exist for $x \simeq a$ but $f(a)$ does not necessarily exist. Context is f and a.

Definition 23
A ***deleted interval of*** a *is an interval around* a *not containing* a.

$$\boxed{\text{The limit of } f \text{ at } a \text{ is the value that } f \text{ should take in order to be continuous at } a.}$$

Definition 24
Let f *be a real function defined on a deleted interval of* a*. Context is* f *and* a*. We say that* f ***has a limit at*** a *if there exists an observable number* L *such that if we had* $f(a) = L$ *then* f *would be continuous at* a*,*

In other terms, if there is an observable number L such that

$$x \simeq a \Longrightarrow f(x) \simeq L.$$

Of course, by this definition, if f *is* continuous at a, then the limit of f at a is $f(a)$.

The definition of limit can also be interpreted in the following way:

If f has a limit at a then it is the observable neighbour of $f(a + dx)$.

If L is the limit of f at a we write

$$f(a + dx) \simeq L$$

or

$$\lim_{x \to a} f(x) = L,$$

Exercise 113

Calculate

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3}.$$

Show that it is equal to

$$\lim_{h \to 0} \frac{2(3 + h)^2 - 7(3 + h) + 3}{(3 + h) - 3}.$$

Exercise 114

Consider the signum function sgn, defined by

$$\text{sgn} : x \mapsto \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

Check that sgn is defined around 0. Does it have a limit at 0?

One Sided Limits

A function is defined **on the left** (respectively **on the right**) of a, if $f(x)$ exists for $x \simeq a$, $x < a$ (respectively $x \simeq a$, $x > a$).

Definition 25

Let f be a real function defined on the left of a. The function f ***has a limit on the left of*** *a if there is an obervable number L such that*

$$x \simeq a \text{ and } x < a \implies f(x) \simeq L.$$

If the limit on the left exists it is unique (it is the observable neighbour of $f(x)$). We write:

$$\lim_{x \to a_-} f(x) = L, \quad \text{or} \quad x \simeq a_- \Rightarrow f(x) = L.$$

The symbol a_- indicates that we choose numbers less than a.

Similarly we define the **limit on the right of** a and write:

$$\lim_{x \to a_+} f(x) = L, \quad \text{or} \quad x \simeq a_+ \Rightarrow f(x) = L.$$

The symbol a_+ indicates that we choose numbers greater than a.

The limit is only a rewriting. The "equal" sign used is there to say that the limit is the value that the function can be ultraclose to.
When a limit appears in a problem, the first thing to do is to rewrite it in terms of ultracloseness.

The symbol "∞" can be used to indicate that the function takes ultralarge values. Since if a function has a maximum, by closure, the maximum would be observable, the fact that it reaches ultralarge values implies that it has no maximum, hence that the intervakl of possible results is infinite.

Practice exercise 15 Answer page 68

Calculate the following limits. The answer should be a number, $+\infty$, $-\infty$ or "does not exist"

(1) $\lim\limits_{x\to\infty} \dfrac{6x-4}{2x+5}$

(2) $\lim\limits_{x\to\infty} x^3-10x^2-6x-2$

(3) $\lim\limits_{x\to\infty} \dfrac{x^2-x+4}{3x^2+2x-3}$

(4) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+2}}{\sqrt{3x+1}}$

(5) $\lim\limits_{x\to\infty} x-\sqrt{x}$

(6) $\lim\limits_{x\to\infty} \sqrt[3]{x+2}$

(7) $\lim\limits_{x\to 0_-} 1+\dfrac{1}{x}$

(8) $\lim\limits_{x\to 0} \dfrac{1}{x^2}-\dfrac{1}{x}$

(9) $\lim\limits_{x\to 0} \dfrac{1+2x^{-1}}{7+x^{-1}-5x^{-2}}$

(10) $\lim\limits_{x\to 2} \dfrac{1-x}{2-x}$

(11) $\lim\limits_{x\to 3_+} \dfrac{x+1}{(x-2)(x-3)}$

(12) $\lim\limits_{x\to 3} \dfrac{x+1}{(x-2)(x-3)}$

(13) $\lim\limits_{x\to 1} \dfrac{3x^2+4}{x^2+x-2}$

(14) $\lim\limits_{x\to 2_+} \dfrac{x^2+4}{x^2-4}$

(15) $\lim\limits_{x\to\infty} \sqrt{x^2+1}-x$

(16) $\lim\limits_{x\to-\infty} \sqrt{x^2+1}-x$

(17) $\lim\limits_{x\to\infty} \sqrt{x^2-3x+2}-\sqrt{x^2+1}$

(18) $\lim\limits_{x\to\infty} \sqrt[3]{x+4}-\sqrt[3]{x}$

11

Curve Sketching

Now the rules of de l'Hospital may be also required. The functions may include any combination of functions studied up to now. Some functions may be difficult.

Sketch the curves of the following.

Practice exercise 16 Answer page 69

- $g_1 : x \mapsto x \ln(x)$
- $g_2 : x \mapsto \dfrac{x}{\ln(x)}$
- $g_3 : x \mapsto \dfrac{e^x}{\ln(x)}$
- $g_4 : x \mapsto \dfrac{e^x}{1 + e^x}$
- $g_5 : x \mapsto \dfrac{1}{1 + e^x}$
- $g_6 : x \mapsto \ln(x^2 + 1)$
- $g_7 : x \mapsto \dfrac{e^x}{x - 2}$
- $g_8 : x \mapsto e^{-x^2}$
- $g_9 : x \mapsto \dfrac{x \cdot e^x}{\ln(x)}$

Answers to Practice Exercises

Answers to practice exercice 15, page 65

(1) 3

(2) ∞

(3) $1/3$

(4) $1/\sqrt{3}$

(5) ∞

(6) ∞

(7) $-\infty$

(8) ∞

(9) 0

(10) does not exist

(11) ∞

(12) does not exist

(13) does not exist

(14) ∞

(15) 0

(16) ∞

(17) 0

(18) $-3/2$

Answers to practice exercice 13, page 55

(1) 1 Use $x = 1 - u^2$.

(2) undefined – for $u > 1$ we have the square root of a negative number.

(3) $\frac{8(\sqrt{2}+1)}{15}$ Use $u = 1 + \sqrt{x}$

(4) $\frac{50}{309}$ Use $u = t^2 + 3$

(5) $\frac{195}{8}$ Use $u = x^2 + 2$

(6) $\frac{2}{45}$ Use $u = 4 - x^3$

(7) $-\sqrt{6} + 2\sqrt{2}$ Use $u = 1 + \frac{1}{t}$

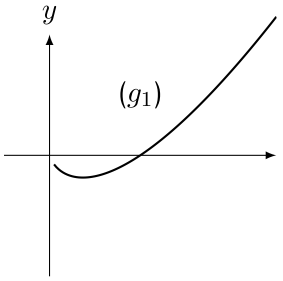
Answers to practice exercice 14, page 58

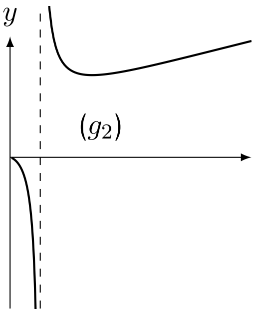
(Integration constant to be added)

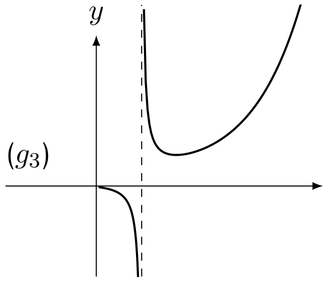
- $F_a : x \mapsto x^5 - x^2 + 4x$
- $F_b : x \mapsto \dfrac{1}{4}x^4 - \dfrac{5}{3}x^3 + \dfrac{3}{2}x^2 - 2x$
- $F_c : x \mapsto x^2 - x$
- $F_d : x \mapsto \dfrac{1}{4}x^5 - \dfrac{1}{4}x^3 + \dfrac{5}{4}x^2 + \dfrac{3}{2}x$
- $F_e : x \mapsto x^2 + x + \dfrac{1}{x}$
- $F_f : x \mapsto 3x - \dfrac{2}{x} + \dfrac{5}{2x^2}$
- $F_g : x \mapsto \dfrac{x^4}{4} - \dfrac{1}{x}$
- $F_h : x \mapsto \dfrac{3}{4}\sqrt[3]{x^4} + \dfrac{3}{2}\sqrt[3]{x^2}$
- $F_i : x \mapsto 2\sqrt{x} + \dfrac{2}{3}\sqrt{x^3}$
- $F_j : x \mapsto \dfrac{1}{3}(x+1)^3$
- $F_k : x \mapsto (3x-2)^5$
- $F_l : x \mapsto \dfrac{1}{8}(2x+1)^4$
- $F_m : x \mapsto -\dfrac{1}{12}(3-x)^{12}$
- $F_n : x \mapsto -\dfrac{1}{20}(3-4x)^5$
- $F_o : x \mapsto \dfrac{2}{9}\sqrt{(3x-2)^3}$
- $F_p : x \mapsto 2\sqrt{x-1}$
- $F_q : x \mapsto -\dfrac{1}{3}(3-x^2)^6$
- $F_r : x \mapsto \dfrac{1}{5}(x^2-3x+1)^5$

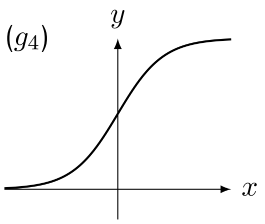
- $F_s : x \mapsto \frac{1}{3}(x^3 - 2x^2 + x - 3)^3$
- $F_t : x \mapsto \frac{2}{3}(4x^2 - 5x)^3$
- $F_u : x \mapsto \frac{1}{8}(3x^2 - 2x + 5)^4$
- $F_v : x \mapsto -\frac{1}{x^2 + 1}$
- $F_w : x \mapsto -\frac{1}{x^2 + x + 3}$
- $F_x : x \mapsto \frac{1}{3}\sqrt{(x^2 + 1)^3}$
- $F_y : x \mapsto 2\sqrt{9 + x^3}$
- $F_z : x \mapsto \frac{2}{3}(x^3 + x + 2)\sqrt{x^3 + x + 2}$
- $F_A : x \mapsto \frac{e^{2x}}{2}$
- $F_B : x \mapsto -\frac{1}{3e^{3x}}$
- $F_C : x \mapsto -\frac{e^{-x^2}}{2}$
- $F_D : x \mapsto -\frac{1}{\ln(2)}2^{-x}$
- $F_E : x \mapsto \frac{1}{3}(e^{2x} + 1)^{\frac{3}{2}}$
- $F_F : x \mapsto e^x(x^2 - 2x + 2)$
- $F_I : x \mapsto \frac{\ln(x + \frac{3}{2})}{2}$
- $F_J : x \mapsto 2x + 2\ln(x - 1)$
- $F_K : x \mapsto x - 2\ln(x + 1)$
- $F_L : x \mapsto 2x\left(\frac{\ln(x)^2}{2} - \ln(x) + 1\right)$
- $F_N : x \mapsto x\ln(x) - x$
- $F_O : x \mapsto x - \ln(x + 1)$
- $F_P : x \mapsto \ln(\ln(x))$

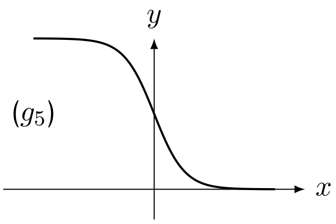
Answers to practice exercice 16, page 67

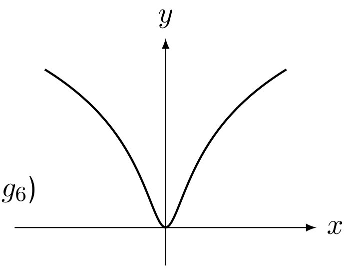


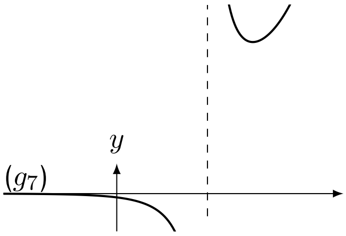












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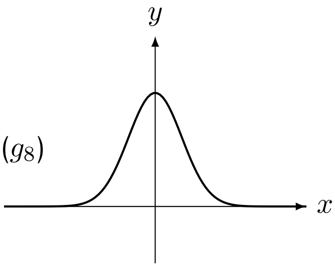

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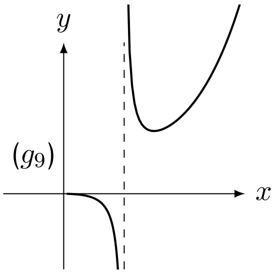

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11.1 Summary of principles

Observability Principle

- A number is observable relative to a context if it is observable relative to at least one parameter of the context.
- Every number is observable relative to some context.
- Two numbers a and b will always have a common context. If a is not observable relative to b, then b will be observable relative to a.

Closure Principle

Numbers, sets or functions, defined without reference to observability are always observable. If an object satisfies a given property, then there is an observable object satisfying that property

Principle of ultrasmallness

Relative to any context, there exist ultrasmall real numbers.

Contextual Notation

The only acceptable properties are those that do not refer to observability or those that use the symbol "$\simeq$".

Principle of the observable neighbour

Relative to a context, any real number x which is not ultralarge can be written in the form $a + h$ where a is observable and $h \simeq 0$.